On the Union-closed conjecture

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1 Introduction

The Union Closed Conjecture sounds simple enough, yet it's surprisingly tricky to crack. Its elusive nature adds to its allure, keeping mathematicians intrigued. We say a family of sets is *union-closed* if the union of any two sets from the family belongs to the family.

Let $[n] = \{1, 2, 3, \ldots, n\}$, and let $\mathcal{F} \subseteq 2^{[n]}$. $\mathcal{F}$ is union-closed if for all $A, B \in \mathcal{F}, A \cup B \in \mathcal{F}$

Union-Closed Conjecture:

> If $\mathcal{F}$ is closed under union, $\exists i \in \{1, 2, 3, \cdots, n\}$ that belongs to atleast half of the sets in $\mathcal{F}$

One obvious question might pop into your mind. What's special about 1/2? It's not hard to notice that the power set satisfies this property, and hence is a natural motivation for formulating the problem this way.

2 Preliminaries

First, we fix some notation.

We now define the *universe* of $\mathcal{F}$. $\mathcal{U}(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} F$. We also define the following:

$$\mathcal{F}_x = \{F \in \mathcal{F} : x \in F\}$$
$$\mathcal{F}_{\bar{x}} = \mathcal{F} \setminus \mathcal{F}_x$$

If $\mathcal{F}$ is union-closed, so are $\mathcal{F}_x$ and $\mathcal{F}_{\bar{x}}$

Intersection Formulation

There are several equivalent formulations of the Union-Closed Conjecture, one of which is the Intersection-Closed Conjecture.

Let $\mathcal{F}$ be any union-closed family of sets. We construct the family $\mathcal{D}$ as follows:

$$\mathcal{D} = \{\mathcal{U}(\mathcal{F}) \setminus F : F \in \mathcal{F}\}.$$

Then $\mathcal{D}$ is said to be an *intersection-closed* family. That is, if $A, B \in \mathcal{D}$, then $A \cap B \in \mathcal{D}$.

Intersection-Closed Conjecture

> If $\mathcal{D}$ is closed under intersection, $\exists i \in \{1, 2, 3, \cdots, n\}$ that belongs to atleast half of the sets in $\mathcal{D}$

Call an element $x \in \mathcal{U}(\mathcal{F})$ abundant if $|\mathcal{F}_x| \geq \frac{1}{2}|\mathcal{F}|$. Analogously, one can define *rare* elements in the intersection-closed formulation: Call an element $y \in \mathcal{U}(\mathcal{D})$ rare of $|\mathcal{D}_y| \leq \frac{1}{2}|\mathcal{D}|$.
It is also safe to make the following assumptions for the sake of this exposition:

1. Let $\mathcal{F}$ be any family of sets. $\mathcal{F}$ necessarily needs to have finitely many elements. Suppose we have a family $\mathcal{F}' = \{k, k+1, k+2, \cdots\}$, where $k \in \mathbb{N}$. Of course, we have no element occurring infinitely many times, hence we restrict our attention to families with finitely many elements.

2. The universe $\mathcal{U}(\mathcal{F})$ is finite as well.

3. Any union closed set can always contain $\emptyset$

Definition. A union closed set $\mathcal{F}$ is said to be *separating* if for any two elements in the family, there exists a member set containing exactly one of them.

3 Some examples:

Firstly, let us take a simple example to see what is being stated here. Consider $\mathcal{P}(S)$, where $S = \{1, 2, 3, 4\}$. Then,

$$\mathcal{P}(S) = \{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}.$$

Consider the element $\{1\}$, it is present in 8 member sets, thus verifying the conjecture. We now state a few applications of the conjecture in several scenarios;

1. Given a finite collection of colors, if every pair of colors can be mixed to produce another color in the collection, then at least half of the colors can be obtained by mixing just two colors from the collection.

2. Let G be a finite simple graph (i.e., the edge set and vertex set are finite sets). If, for every pair of edges a and b , there exist a vertex v such that vertex v is incident to either edge a or b, then there exist a vertex v incident to atleast half of the edges of G.

3. Consider a positive integer n and the family of all divisors of n, including 1 and n itself. If this family is closed under taking unions, then there exists a divisor of n that divides at least half of the divisors of n.

4. Let G be a finite graph, and consider the family of all subgraphs of G, including G itself and the empty graph. If this family is closed under taking unions of subgraphs, there exists a vertex in G that appears in at least half of the subgraphs in the family.

5. Given a finite group G, is there an element of prime power order which is contained in at most half the subgroups of G?

6. Let G be a finite solvable group. Then there is an element whose order is a prime power that is contained in at most half the subgroups of G.

4 A Brief History

The conjecture is generally attributed to Frankl, and hence it is also called as Frankl's Conjecture. Several mathematicians, however, also consider it to be well-known and a 'folklore'. Regardless, the conjecture has spread far and wide, with a diverse group of mathematicians applying a myriad of techniques to tackle this seemingly simple problem. There was also a PolyMath project initiated by Prof. Timothy Gowers, a famed combinatorialist , which introduced several strengthenings to the conjecture, as well as showed some to be false. We will look into a few of these techniques in this report. There are a couple of observations that have helped progress in resolving this conjecture:

5 Where's the Bottleneck?

The simple formulation of the problem is one of the primary reasons why it is hard to prove it: there are hardly any efficient techniques compared to brute force. An induction based technique wouldn't work, mostly because we lack the one thing induction loves to exploit: structure. There are three main techniques that are listed in this survey by Bruhn and Schaudt [20]: injections, local configurations (Special cases that exploit structure) and averaging.

5.1 Injection

Injection is a comparatively simple technique: We define an injection using the sets defined in Preliminaries. The main idea to note here is that the singleton is always an abundant element.

Let $\mathcal{F}$ be a family of sets and x be an element such that $\{x\} \in \mathcal{F}$. We define the following injective map:

$$\begin{gathered} \phi : \mathcal{A}_{\bar{x}} \longrightarrow \mathcal{A}_x \\ A \mapsto A + x \end{gathered}$$

It is routine verification to check that this is injective. Then,

$$2|\mathcal{A}_x| \geq |\mathcal{A}_x| + |\mathcal{A}_x| = |\mathcal{A}|.$$

And hence, x is an abundant element. Such injections have been used to show that the conjecture holds for several special cases, including that of chordal bipartite graphs.

However, trying to generalise this, we fail very quickly: although 2 element sets have atleast one abundant element, one can define a family of 3 element sets that has a non abundant element. From there, we have managed to explicitly construct families with k-element sets that do not have an abundant element. Thus, we are not able to predict where an abundant element might occur!

5.2 Special Cases

The next idea is to exploit the particular structure of certain subcases to prove the conjectures for these cases alone. As we observed earlier, we are on the lookout for abundant elements. We have the following result due to Poonen [6]:

Theorem. Let $\mathcal{F}' \subseteq [k]$ be a union closed family. Then every union closed family $\mathcal{F}$ such that $\mathcal{F}' \subseteq \mathcal{F}$ satisfies the conjecture. In particular, we can find an abundant element of $\mathcal{F}$ in $[k]$.

A further generalisation of this result was used to prove the case of subcubic graphs. The conjecture has been verified for families with either few member sets or few elements.

5.3 Averaging

Poonen [6] conjectured the following, which serve as the backbone for the averaging paradigm:

Conjecture. Let $\mathcal{F}$ be a separating union closed family. If $\mathcal{F} \neq 2^{[n]}$, it contains an element that appears in strictly more than half the members of $\mathcal{F}$.

He also gave the following conjecture about the unique abundant element:

Conjecture. Let $\mathcal{F}$ be a separating union closed family. If $\exists f \in \mathcal{F}$ such that a is abundant (and unique), then

$$\mathcal{F} = \{\emptyset\} \cup \{G + f : G \subseteq \mathcal{U}(\mathcal{F}) - f\}$$

An averaging technique can be used to overcome the obstacle of looking for an abundant element. We have the following result:

Theorem. Let $\mathcal{F}$ be a family of sets, and let $\mathcal{U}(\mathcal{F})$ be its universe, as discussed earlier. Then, the conjecture holds if

$$\tfrac{1}{|\mathcal{F}|} \cdot \textstyle\sum_{F \in \mathcal{F}} |F| \geq \tfrac{1}{2} |\mathcal{U}(A)|$$

The average set size of any member of the family should be atleast half the universe size, for the conjecture to hold. This is used in a few results, the most important of which is below.

Theorem (Nishimura and Takahashi [1])**.** Let the size of the universe be m. Let $\mathcal{F}$ be a union-closed family with more than $2^m - \frac{1}{2}\sqrt{2^m}$ member sets. Then $\mathcal{F}$ satisfies the Frankl conjecture.

We have this nice observation noted by Shayan Oveis Gharan during an expository talk.

Theorem. If $\mathcal{F}$ is union closed, $\exists i \in \{1, 2, 3, .., n\}$ that belongs to atleast $\frac{1}{\sqrt{2n}}$ fraction of non-empty sets in $\mathcal{F}$.

Theorem (David Reimer [2])**.** If $\mathcal{F}$ is union closed, $\sum_{F\in\mathcal{F}}[|F|] \geq \frac{log_2(|\mathcal{F}|)}{2} \cdot |\mathcal{F}|$

In particular, $\exists i \in \{1, 2, \cdots, n\}$ that belongs to atleast $\frac{log_2(|\mathcal{F}|)}{2}$ fraction of sets in $\mathcal{F}$. A few bounds stronger than Reimer's have been found, but this pretty much does the job for us. To conclude this section, we state the following result:

Theorem. Let $\mathcal{F}$ be a separating family on m elements. If $\mathcal{F}$ has at most 2m member sets, then it satisfies the Frankl conjecture.

This result is due to a theorem by Falgas-Ravry [3]

A lot of work has been done in the averaging case. For more such results, we direct the reader to the work of Reimer [2]

6 Classes for which the Conjecture holds

One natural way of attacking any problem is by finding smaller subclasses for which the conjecture holds . Let $\mathcal{F} \subseteq 2^{[n]}$ be any union-closed family of sets. There has been some progress when we make some strong assumptions about $\mathcal{F}$. Let $k = |\mathcal{F}|$, and let $m = |\mathcal{U}(\mathcal{F})|$, where $\mathcal{U}(\mathcal{F}) = \cup_{F\in\mathcal{F}} F$

- Zivkovic and Vuckovic [7] demonstrated that $\mathcal{F}$ has atmost 12 elements or at most 50 member sets.
- If $\mathcal{F}$ is separating, $n \leq 2m$
- Balla, Bollabas, and Eccles [4] have shown that the conjecture holds when $|\mathcal{F}| \geq \frac{2}{3}2^n$.
- Karpas improved this by showing that it holds for $|\mathcal{F}| \geq 2^{n-1}$

One of the best known lower bounds for a long time was due to Knill.

Theorem (Knill [9])**.** Any union closed family $\mathcal{F} \subseteq 2^{[n]}$ has an element of frequency atleast $\frac{n-1}{log_2(n)}$

Wojick [10] improved this bound to $\frac{2.4}{log_2(n)}$. Asymptotically, one can just say that there is an element contained in $\Omega(\frac{|\mathcal{F}|}{log_2(|\mathcal{F})|})$

We have also managed to make use of bounds on the size of the universe itself.

Theorem (Lo Faro [8])**.** Suppose Frankl's Conjecture fails. Let $\mathcal{F}$ be one such counterexample. Let $m = min\,\{|\mathcal{U}(\mathcal{F}|)\}$, the minimum over all such counterexamples. Then any counterexample has atleast $4m - 1$ member sets.

7 Graph Formulation

One can formulate the Union-Closed conjecture in the context of graphs as well. An independent set of a graph G is a subset $S \subseteq V$ of the vertex set V such that no two vertices in the set are adjacent. An independent set in which no vertex of G can be added without violating this condition is said to be maximal. The below conjecture and following results are due to Bruhn et. al [12]. We restate the conjecture as:

> Any bipartite graph with at least one edge contains in each of its vertex sets a vertex that lies in at most half of the maximal independent sets

This formulation can be shown equivalent to the Intersection-Closed conjecture stated earlier. Although admittedly this doesn't look very nice, this gives some much needed structure to the conjecture, which until now we have only viewed from a set theoretic lens.

The above conjecture has been verified for several classes of graphs, which we simply state below without much exposition.

1. A chordal bipartite graph is one in which every cycle of length 6 has a chord (an edge that is not part of the cycle but connects two vertices of the cycle). Chordal Bipartite Graphs satisfy Frankl's conjecture.

2. A bipartite graph in which the occurrence of an edge between two vertices of the vertex sets is determined randomly with some probability $p \in (0, 1)$ is called a random bipartite graphs. Suppose p is fixed. Then, $\forall \delta > 0$ almost every random bipartite graph satisfies Frankl's conjecture upto δ.

3. If every vertex of a graph has degree atmost 3, call it subcubic. Subcubic graphs satisfy Frankl's conjecture.

8 Lattice Formulation

The conjecture can also be reframed in terms of lattices. We recall some basic order theory. A poset $(L, \leq)$ is called a lattice if every 2-element subset of L has a unique greatest lower bound (call this the meet), and a unique least upper bound (call this the join). A simple example would be the collection of all subsets of a set A ordered using the subset inclusion. From the Hasse Diagram, we figure out the infimum is the set intersection operation, whereas the supremum is the set union operation. In this section we consider only finite lattices.

> Let L be a finite lattice with atleast 2 elements. Then $\exists x \in L$ such that x is not the join of any two smaller elements in L. Moreover, the number of elements greater than or equal to x (according to the order) is at most half of the (cardinality of) the lattice.

It was showed that this conjecture is equivalent to the Union Closed Conjecture. This formulation is important: we have a very useful tool- the inclusion operator between sets. This also gives us ideas to think of more special cases.

This was then used to show that certain subclasses of lattices satisfy the union-closed conjecture. Poonen [6] and Rival [16] showed that it holds for geometric lattices. In the earlier days, most of these were not proofs but verifications of special cases. Abe and Nakano [18] showed that the conjecture holds for certain classes called modular lattices. Reinhold [19] generalised this further to lower semimodular lattices. However, most types of lattices remain unconquered.

9 Salzborn Formulation

This is an equivalent formulation of the union-closed sets conjecture due to Salzborn. The main advantage of this formulation is that it only concerns a subclass of union-closed families, which we call normalized.
Note:

1. $\mathcal{A}$ must have at least $|U(\mathcal{A})|$ non-empty sets to separate all elements of its universe.
2. If $\emptyset \in \mathcal{A}$, then $\mathcal{A}$ will have at least $|U(\mathcal{A})| + 1$ sets.

Definition. A union closed family $\mathcal{N}$ is called normalized if $\emptyset \in \mathcal{N}$, $\mathcal{N}$ is separating and $|U(\mathcal{N})| = |\mathcal{N}| - 1$.

Conjecture (Salzborn [11]). Any normalized family $\mathcal{N} \neq \{\emptyset\}$ contains a basis set B with $|B| \geq \frac{1}{2}|\mathcal{N}|$.

It was shown by Salzborn [11] and Poonen [6] that this is indeed equivalent to the Union-Closed Conjecture.

10 Recent Progress!

More recently, there has been progress on finding a constant lower bound for this conjecture. Gilmer [15] uses methods from Information Theory- particularly several equalities pertaining to entropy, to show a constant lower bound. He showed that for any union closed family $\mathcal{F} \subseteq 2^{[n]}, \exists i \in [n]$ which is contained in a 0.01 fraction of the sets in $\mathcal{F}$. This was one of the first *constant* lower bounds to be known. Of course, we are familiar with with the best prior bound given by Knill and Wojick [9, 10]Although the bound of 0.01 is much further than the 0.5 that we need, this is progress in the right direction, using techniques that hadn't been used prior. Soon after, several groups of Mathematicians improved on conjectures proposed by Gilmer in this paper. Will Sawin [13] improved the bound from 0.01 to $\frac{3-\sqrt{5}}{2}(\approx 0.38)$. Chase and Lovett [14] pushed this bound to *approximate* union-closed systems, where for *approximately* all pairs of sets their union belongs to the system, and showed that it is optimal for such systems.

11 Where to from here?

The Union Closed Conjecture is one of the foremost problems that stands testament to the true beauty and versatility of mathematics: the problem is simple to state, and techniques from several areas of mathematics have been used to resolve several cases of the conjecture. One can predict that there is promise from employing techniques from information theory for resolving this conjecture. Another possible direction is the usage of techniques from pseudorandomness and expander graphs: this is motivated from the case of random bipartite graphs. There is another reformulation by El Zahar [17] in the context of hypergraphs. More recently, several proofs in additive combinatorics and number theory are being verified using proof assistants like Lean. Earlier too, proofs for some subcases of the Frankl conjecture was verified using Isabelle, another automated theorem prover. One can harness the power of such proof assistants for further progress. Another possible direction that is not being looked at as much as it should be, is the group theoretic perspective: we have laid out the formulation in the Examples section. It can be shown that the conjecture holds for all finite solvable groups. Looking at the problem from the perspective of finite simple groups (or any algebra, really) might lend a fresh pair of eyes.

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